Low-Degree Hardness of Random Optimization Problems

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Waterloo

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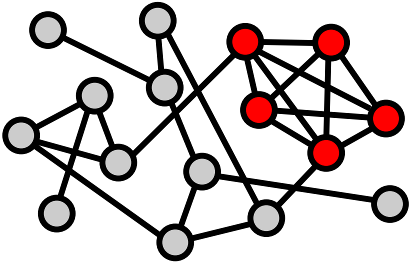
- Max clique in a random graph



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Q: In cases where it seems hard to reach a particular objective value, can we understand why? In a unified way?

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Example (max independent set): given sparse graph $G(n, d/n)$,

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Local algorithms achieve value ALG **and no better**

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Approximate message passing (AMP) algorithms achieve value ALG **and no better** [El Alaoui, Montanari, Sellke '20]

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Solution: lower bounds against a larger class of algorithms (low-degree polynomials) that contains both local and AMP algorithms

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- Power iteration: $Y^k \mathbf{1}$
- Approximate message passing
- Local algorithms on sparse graphs
- Or any of the above applied to $\tilde{Y} = g(Y)$

Planted Problems

For problems with a planted signal, the low-degree framework is already well-established

[Barak, Hopkins, Kelner, Kothari, Moitra, Potechin '16]

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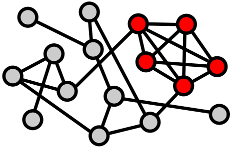
Example (planted clique): $G(n, 1/2)$ with planted k-clique

- Detection
- Recovery



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"Robustness": Gaussian elimination for XOR-SAT

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For some $\epsilon > 0$, no $f : \mathbb{R}^{\otimes p} \to \mathbb{R}^n$ of degree $\mathrm{polylog}(n)$ achieves both of the following with probability $1 - \exp(-n^{\Omega(1)})$:

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Forthcoming: improve $1 + \frac{1}{\sqrt{2}} \to 1 + \epsilon$ (optimal)

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- Overlap gap property (OGP)
  [Gamarnik, Sudan '13]
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Then for any $t \geq (6e)^D$,

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Low-Degree Polynomials are Stable

Theorem

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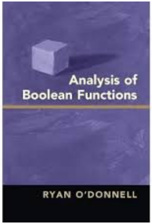
Proof: low-degree polynomials have

- Low noise sensitivty
- Low total influence
- Hypercontractivity



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With non-trivial probability (over path), f's output is "smooth"

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Proof: first moment method [Gamarnik, Sudan '13]

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$$Y^{(0)} \quad Y^{(1)} \quad Y^{(2)} \quad \dots \quad Y^{(m-1)} \quad Y^{(m)}$$

there is no occurrence of

- S independent set in $Y^{(i)}$
- T independent set in $Y^{(j)}$
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Contradicts OGP

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- Connections between heuristics
  - OGP $\to$ Low-Degree

References for the Low-Degree Framework

- Detection (survey article)
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  Kunisky, W., Bandeira
  arXiv:1907.11636

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