Low-Degree Hardness of Random Optimization Problems

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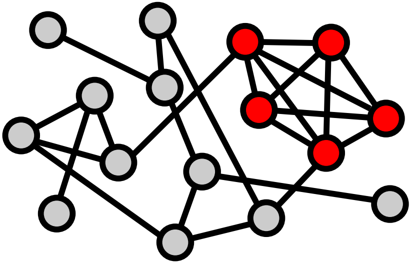
- Max clique in a random graph



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Q: In cases where it seems hard to reach a particular objective value, can we understand why? In a <u>unified</u> way?

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Example (max independent set): given sparse graph $G(n, d/n)$,

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Local algorithms achieve value ALG **and no better**
[Gamarnik, Sudan '13; Rahman, Virág '14]

Spherical Spin Glass

Example (spherical p-spin model): for $Y \in (\mathbb{R}^n)^{\otimes p}$ i.i.d. $\mathcal{N}(0,1)$,

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Approximate message passing (AMP) algorithms achieve value ALG_p **and no better** [El Alaoui, Montanari, Sellke '20]

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Solution: lower bounds against a larger class of algorithms (low-degree polynomials) that contains both local and AMP algorithms

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- Power iteration: $Y^k \mathbf{1}$
- Approximate message passing
- Local algorithms on sparse graphs
- Or any of the above applied to $\tilde{Y} = g(Y)$

Planted Problems

Low-degree algorithms are already well-studied for problems with a planted signal

[Barak, Hopkins, Kelner, Kothari, Moitra, Potechin '16]

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<u>This work</u>: extend low-degree framework to non-planted setting

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Theorem [Gamarnik, Jagannath, W. '20] Let $p \geq 4$ be even.
For some $\epsilon > 0$, no $f : (\mathbb{R}^n)^{\otimes p} \to \mathbb{R}^n$ of degree $\mathrm{polylog}(n)$ achieves both of the following with probability $1 - \exp(-n^{\Omega(1)})$:

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- Objective: $H(f(Y)) \geq \text{OPT} - \epsilon$
- Normalization: $\|f(Y)\| \approx 1$

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- $\{i : f_i(Y) \in [2/3, 1]\}$ is a near-indep set of size $(1+\epsilon)\frac{\log d}{d} n$

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- Stability of low-degree polynomials
- Overlap gap property (OGP)
  [Gamarnik, Sudan '13]
  [Rahman, Virág '14]
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With non-trivial probability (over path), f's output is "smooth"

Overlap Gap Property [Gamarnik, Sudan '13]

Overlap gap property (OGP): with high probability over $Y \sim G(n, d/n)$, there does not exist $S, T \subseteq [n]$ such that

- S, T independent sets
- $|S|, |T| \geq (1 + \frac{1}{\sqrt{2}})\Phi \qquad \Phi := \frac{\log d}{d} n$
- $|S \cap T| \approx \Phi$

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Proof: first moment method [Gamarnik, Sudan '13]

Ensemble OGP [CGPR'17, GJ'19]

Ensemble OGP: with high probability over

$$Y^{(0)} \quad Y^{(1)} \quad Y^{(2)} \quad \ldots \quad Y^{(m-1)} \quad Y^{(m)}$$

there does not exist $S, T \subseteq [n]$ such that

- S independent set in some $Y^{(i)}$
- T independent set in some $Y^{(j)}$
- $|S|, |T| \geq (1 + \frac{1}{\sqrt{2}})\Phi \qquad \Phi := \frac{\log d}{d} n$
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Suppose $f(Y)$ outputs independent sets of size $(1 + \frac{1}{\sqrt{2}})\Phi$

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Contradicts OGP

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Forbidden structure: with high probability over

$$Y^{(0)} \quad Y^{(1)} \quad \dots \quad Y^{(Lm)} \qquad (L \approx 1/\epsilon^2)$$

there does not exist $S_0, \dots, S_L \subseteq [n]$ such that

- S_k independent set in some $Y^{(t_k)}$
- $|S_k| \geq (1+\epsilon)\Phi$
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  - After m steps, algorithm's output must change

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- Proof of OGP for p-spin (for $p \geq 4$ even)
  [Chen, Sen '15; Auffinger, Chen '17]

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- Connections between heuristics
  - OGP $\rightarrow$ Low-Degree

References for the Low-Degree Framework

- Detection (survey article)
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  Kunisky, W., Bandeira
  arXiv:1907.11636

- Recovery
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  Gamarnik, Jagannath, W.
  arXiv:2004.12063