Counterexamples to the Low-Degree Conjecture

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Joint work with:

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High-Dimensional Testing Problems

Example: planted clique

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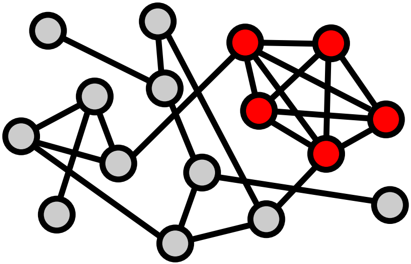
Distinguish between
$\mathbb{Q}$: random graph $G(n, 1/2)$
$\mathbb{P}$: random graph $G(n, 1/2)$ with planted k-clique
with error probability $o(1)$



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Other examples:

- Sparse PCA
- Planted CSPs
- Community detection in graphs
- Tensor PCA
- Spiked matrix models
- ...

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- Low-degree polynomials

Low-Degree Algorithms

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Inspired by a line of work on sum-of-squares:

[Barak, Hopkins, Kelner, Kothari, Moitra, Potechin '16]

[Hopkins, Steurer '17]

[Hopkins, Kothari, Potechin, Raghavendra, Schramm, Steurer '17]

[Hopkins '18] (PhD thesis)

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Low-degree algorithm: multivariate polynomial
$f : \mathbb{R}^N \to \mathbb{R}$ e.g. $N = \binom{n}{2}$
"Low-degree": degree $O(\log n)$

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- E.g. planted clique: low-degree algorithms succeed when $k \gg \sqrt{n}$ and fail when $k \ll \sqrt{n}$

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How to formalize "natural"?

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Conjecture (Hopkins '18): Suppose that

- $\mathbb{Q}$ has i.i.d. entries,
- $\mathbb{P}$ is "symmetric",
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Symmetry: typical for high-dimensional problems, but is this assumption needed?

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This refines our understanding of what types of problems we expect low-degree algorithms to be optimal for

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  - Degree-D polynomials cannot distinguish $\mathbb{Q}$ from $\mathbb{P}$
- Poly-time decoding algorithm can recover $c \in \mathcal{C}$ after a random δ fraction of bits are flipped
  - Gives a poly-time algorithm to distinguish $\mathbb{Q}$ from $T_\delta \mathbb{P}$

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For more on the low-degree method (survey):

- **Notes on Computational Hardness of Hypothesis Testing: Predictions using the Low-Degree Likelihood Ratio**
  Kunisky, W., Bandeira
  arXiv:1907.11636