Optimality of AMP Among Low-Degree Polynomials

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Spiked Wigner Model

signal-to-noise ratio $s > 0$

$$Y = \frac{s}{\sqrt{n}} \theta \theta^\top + Z$$

observed data, n-by-n matrix

rank-1 “signal”

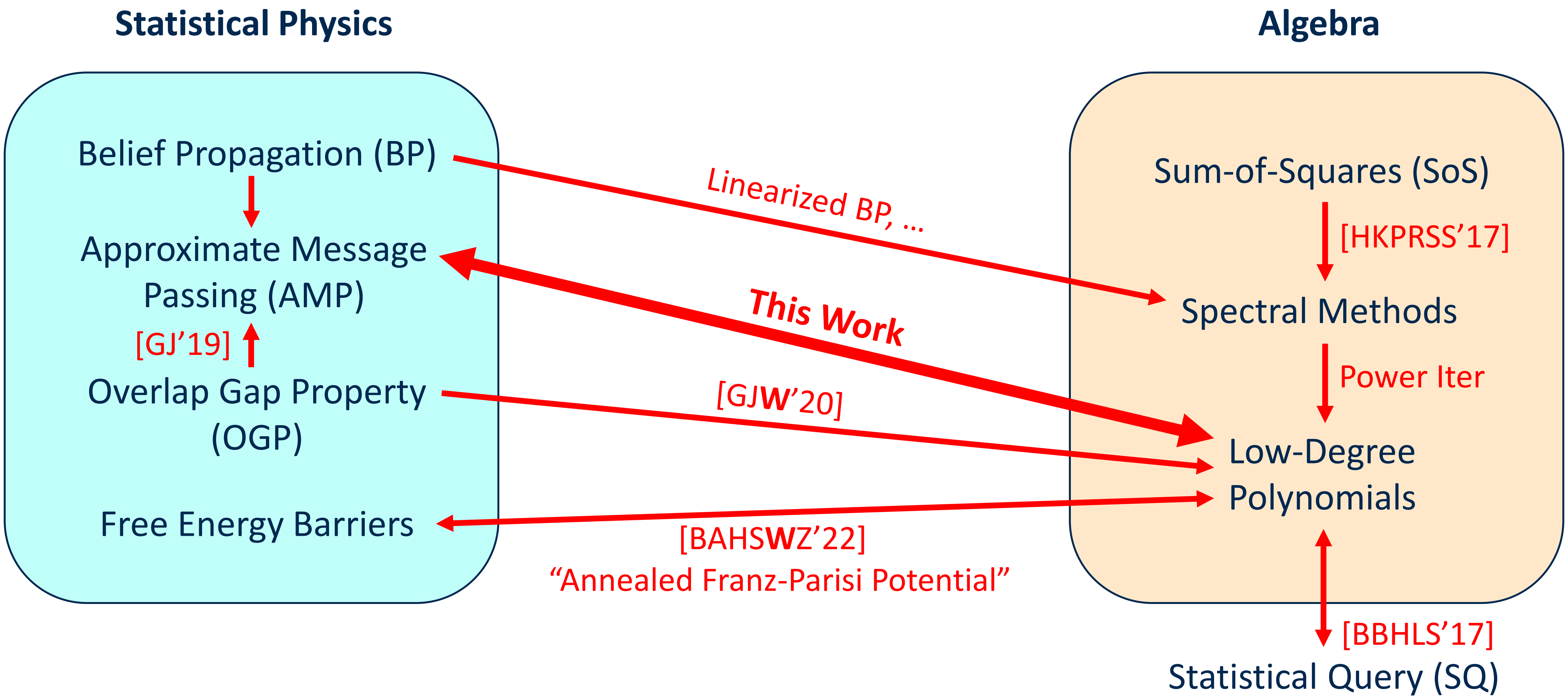
iid Gaussian “noise”

θ – unknown vector with entries iid from known fixed prior π

Goal: given Y, estimate θ

Simple “signal plus noise” model, testbed

What Are The Best Algorithms?



AMP for Spiked Wigner Model

$$Y = \frac{s}{\sqrt{n}}\theta\theta^\top + Z$$

Iterate

estimation accuracy

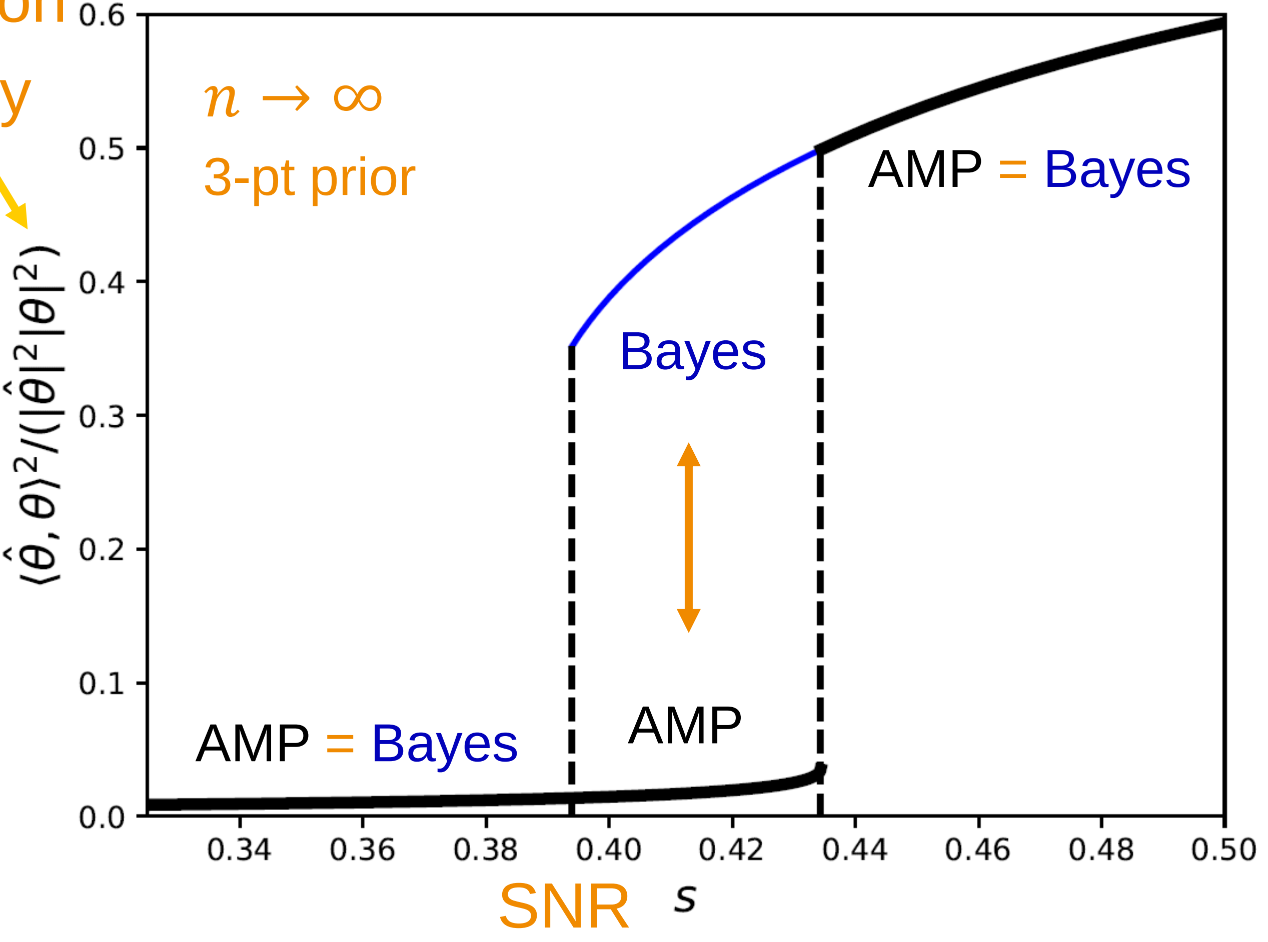
$$x^0 = 0$$

vector, estimate for θ

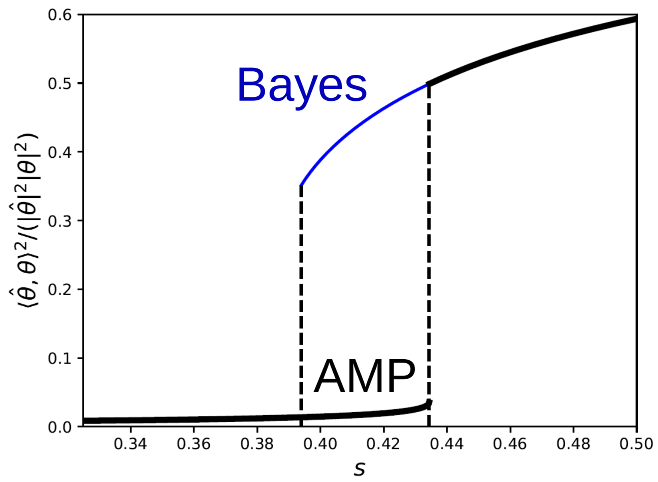
$$x^t = \frac{1}{\sqrt{n}} Y f(x^{t-1}) - \mathrm{b}_t f(x^{t-2})$$

entrywise transform

Onsager term



Bayes is not comp. efficient, gap



Main Result

Conjecture

AMP has optimal MSE among all poly-time algorithms

Theorem *(Montanari,* ***W*** *'22)*

AMP has optimal MSE among all constant-degree polynomials

AMP (with const num iter) takes the form $(\hat{\theta}_1(Y), \ldots, \hat{\theta}_n(Y))$ where $\hat{\theta}_i$ is a const-deg multivariate polynomial in the entries of Y

We show AMP is the best estimator of this form; sharp constant

Comments

Biased prior: $\mathbb{E}[\pi] \neq 0$

Open: mean-zero prior π, $O(\log n)$ iterations/degree

Open: rule out higher degree polynomials

conjecture: need degree $n^{1-o(1)}$ to beat AMP

AMP is sub-optimal for tensor PCA [Montanari,Richard'14]

Kikuchi hierarchy "redeems" physics [**W**,Alaoui,Moore'19]

Proof suggests how to test if AMP is optimal for a given problem

Low-Degree Estimation Lower Bounds

Given Y, estimate θ_1

Predictor: $\mathrm{MMSE}_{\leq D} \coloneqq \inf_{p \deg D} \mathbb{E}[(p(Y) - \theta_1)^2]$

- Planted submatrix, planted dense subgraph [Schramm,**W**'20]
- Hypergraphic planted dense subgraph [Luo,Zhang'20]
- Tensor decomposition [**W**'22]

This work: exact value of $\lim_{D\to\infty} \lim_{n\to\infty} \mathrm{MMSE}_{\leq D}$

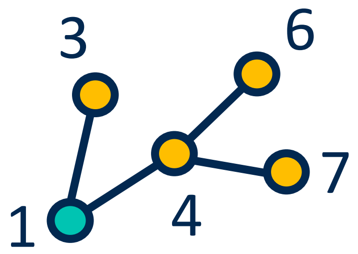
$$Y = \frac{s}{\sqrt{n}} \theta\theta^\top + Z$$

Proof Sketch: AMP vs Low-Deg

I. AMP is as powerful as any "tree-shaped" polynomial

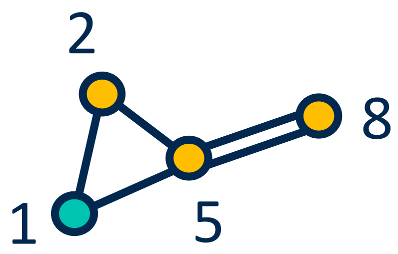
II. Tree-shaped polynomials are as powerful as all polynomials (of the same degree)

tree



$$f(Y) = Y_{13}Y_{14}Y_{46}Y_{47}$$

non-tree



$$g(Y) = Y_{12}Y_{15}Y_{25}Y_{58}^2$$

$$Y = \frac{s}{\sqrt{n}} \theta\theta^\top + Z$$

I. AMP vs Tree Polynomials

Claim: $\lim_{t\to\infty} \lim_{n\to\infty} \mathrm{MSE}_t^{\mathrm{AMP}} = \lim_{D\to\infty} \lim_{\mathrm{n}\to\infty} \mathrm{MMSE}_{\le D}^{\mathrm{Tree}}$

($\geq$) AMP is a tree polynomial

($\leq$) Consider the best tree polynomial, WLOG symmetric

Given any symmetric const-deg tree polynomial, can construct a “message-passing” (MP) scheme to compute it

Prior work: AMP has best MSE among all MP schemes

[Celentano,Montanari,Wu’20; Montanari,Wu’22]

$$Y = \frac{s}{\sqrt{n}}\theta\theta^\top + Z$$

II. Tree Poly vs All Poly

Remains to prove: $\lim_{n\to\infty} \mathrm{MMSE}^{\mathrm{Tree}}_{\le D} = \lim_{n\to\infty} \mathrm{MMSE}_{\le D}$ (rest of talk)

Conclude:

$$\lim_{t\to\infty}\lim_{n\to\infty} \mathrm{MSE}^{\mathrm{AMP}}_{t} = \lim_{D\to\infty}\lim_{n\to\infty} \mathrm{MMSE}^{\mathrm{Tree}}_{\le D} = \lim_{D\to\infty}\lim_{n\to\infty} \mathrm{MMSE}_{\le D}$$

AMP Tree Poly All Poly

$$Y = \frac{s}{\sqrt{n}} \theta\theta^\top + Z$$

II. Tree Poly vs All Poly

Remains to prove: $\lim_{\mathrm{n}\to\infty} \mathrm{MMSE}_{\leq D}^{\mathrm{Tree}} = \lim_{\mathrm{n}\to\infty} \mathrm{MMSE}_{\leq D}$

$$\mathrm{MMSE}_{\leq D} \coloneqq \inf_{p \deg D} \mathbb{E}[(p(Y) - \theta_1)^2] = \mathbb{E}[\theta_1^2] - c^\top M^{-1} c$$

where:

$\{H_A\}$ – basis for (symmetric) const-deg polynomials

$c_A \coloneqq \mathbb{E}[H_A(Y) \cdot \theta_1]$ $\qquad$ $M_{AB} \coloneqq \mathbb{E}[H_A(Y) \cdot H_B(Y)]$

(see board)

Summary

Equivalence of constant-iter AMP and constant-degree polynomials in the spiked Wigner model with any fixed prior

AMP = tree polynomials = all polynomials

Key property of Wigner model for "tree = all": block diagonal

use this to test if AMP is optimal for a given problem?

Thanks!