Low-Degree Polynomials: Overview and Recent Developments

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New survey on arXiv, “Computational Complexity of Statistics: New Insights from Low-Degree Polynomials”

arXiv:2506.10748

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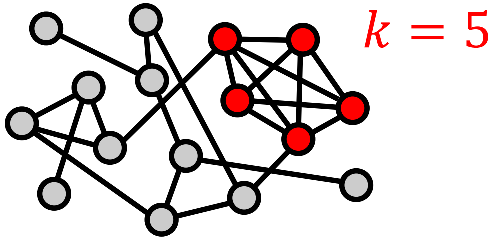

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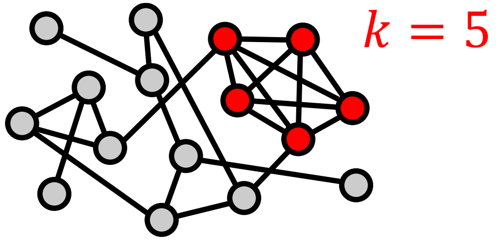

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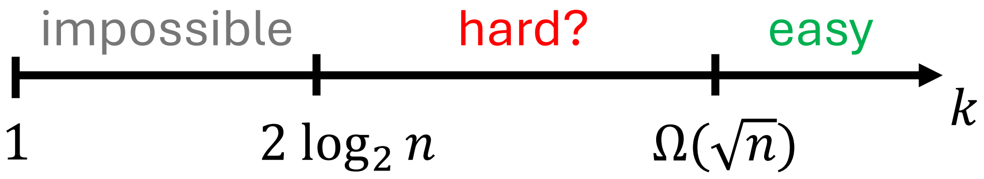

- Statistical-computational gap
  - Impossible: Any estimator fails [Arias-Castro, Verzelen ‘14]
  - Easy: Poly-time algorithm exists [Alon, Krivelevich, Sudakov ‘98]
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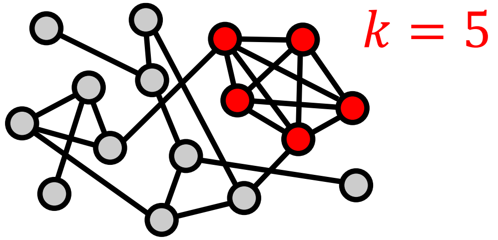

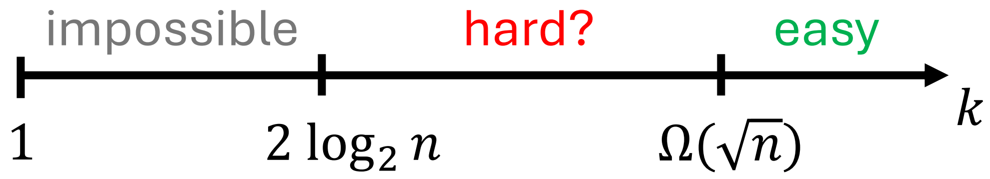

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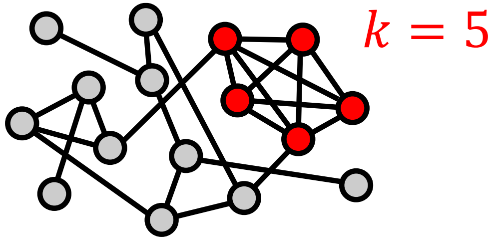

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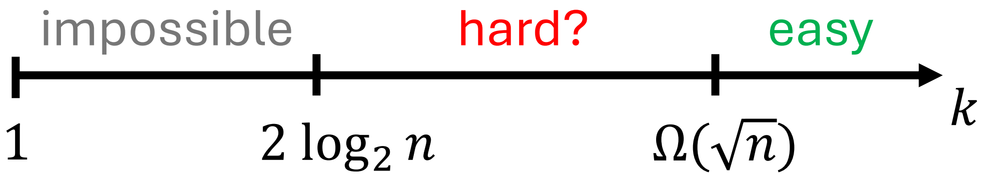

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- Other examples: sparse PCA, community detection, clustering, …

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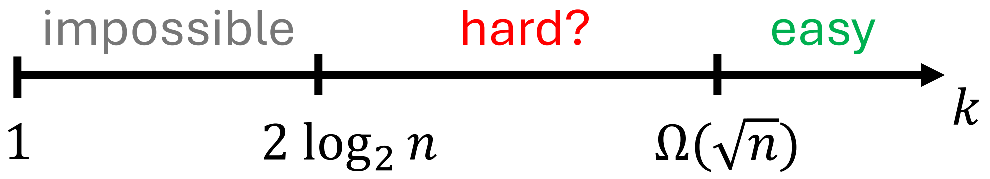

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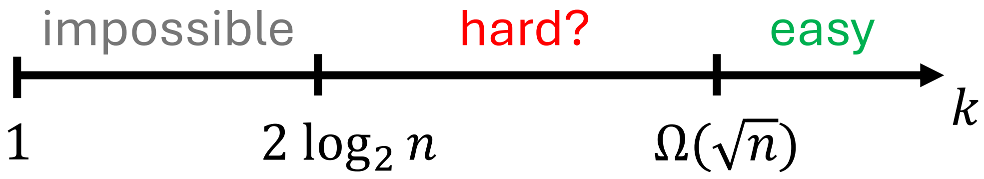

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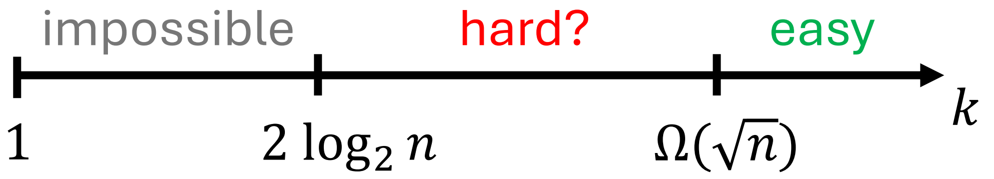

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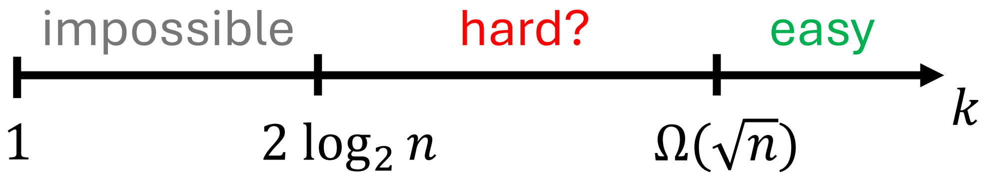

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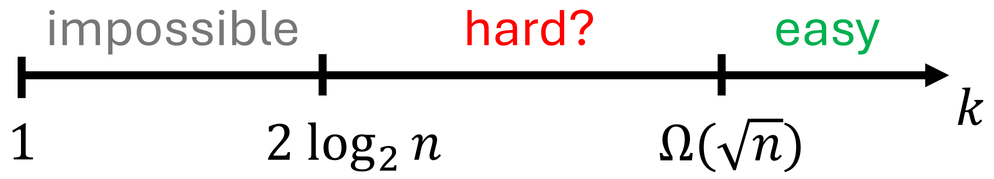

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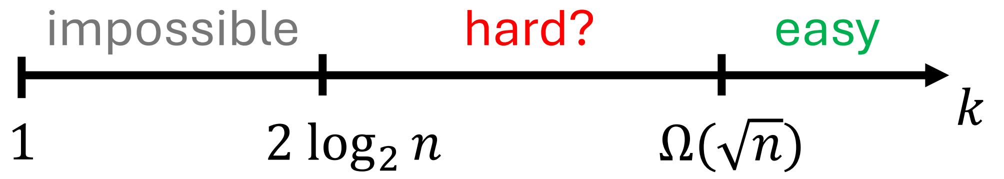

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- Instead, various approaches to average-case hardness:
  - Conditional hardness via reductions
    - Popular starting assumptions: planted clique conjecture, shortest vector on lattices, …
  - Unconditional failure of restricted classes of algorithms
    - Sum-of-squares hierarchy (SOS)
    - Statistical query model (SQ)
    - Approximate message passing (AMP)
    - Overlap gap property (OGP)
    - …
    - Low-degree polynomials (main focus)

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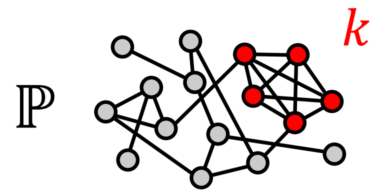

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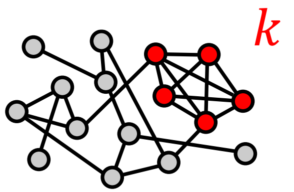

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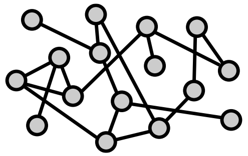

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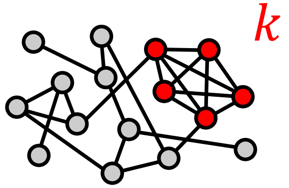

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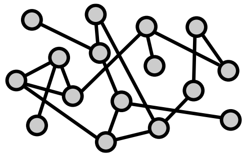

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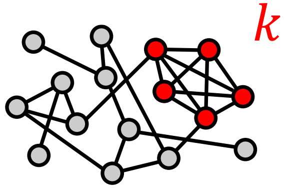

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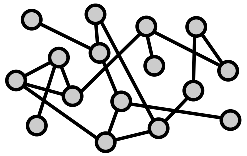

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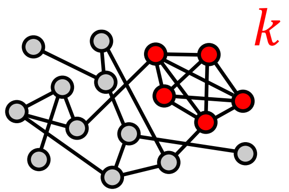

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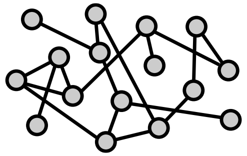

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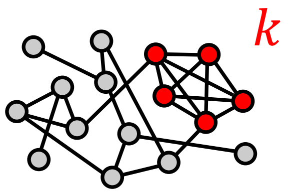

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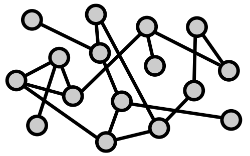

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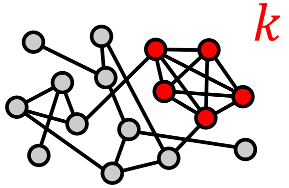

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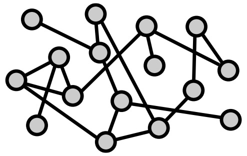

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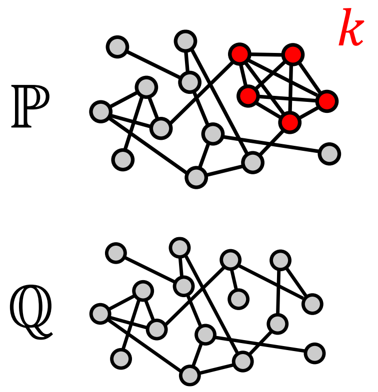

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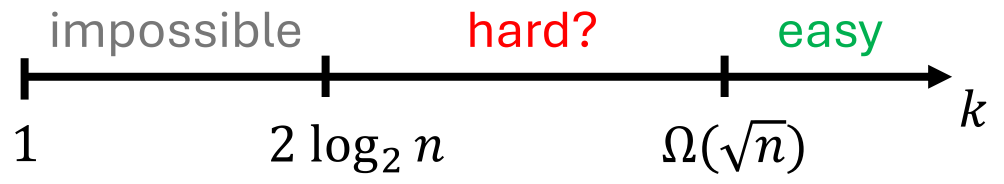
- For planted clique, both have the same thresholds



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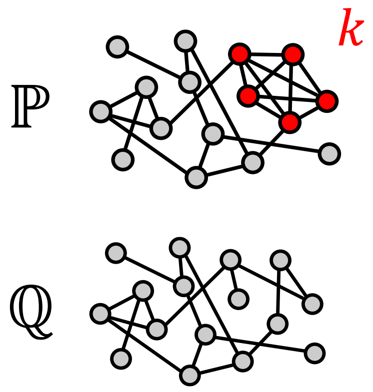

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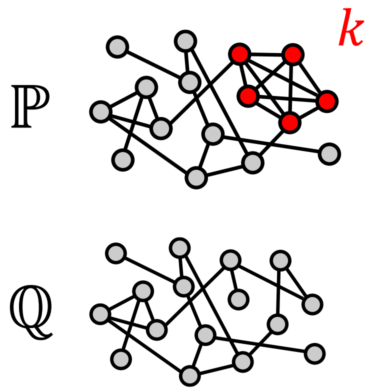

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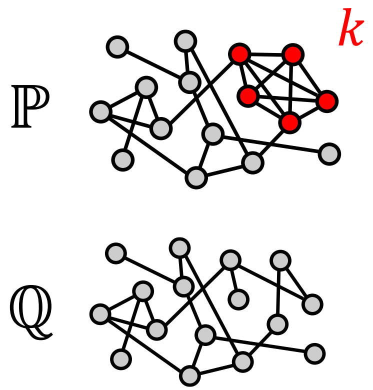

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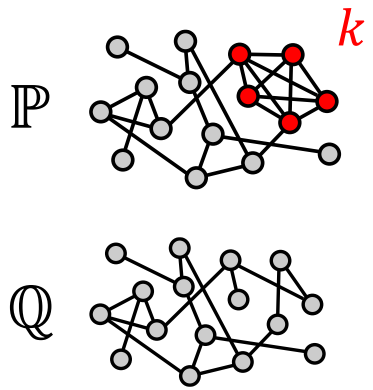

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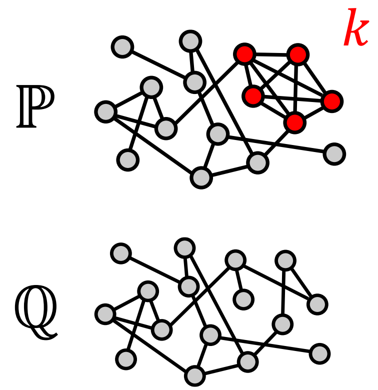

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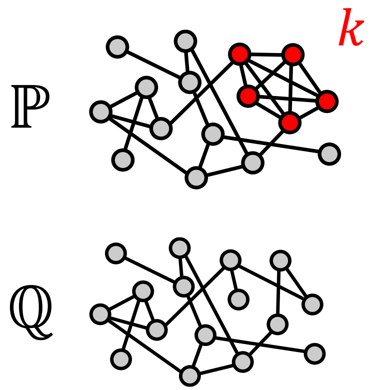

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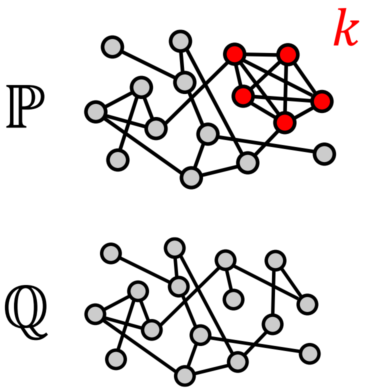

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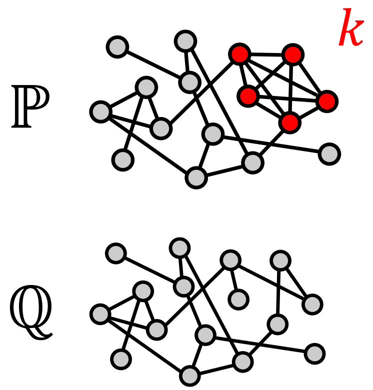

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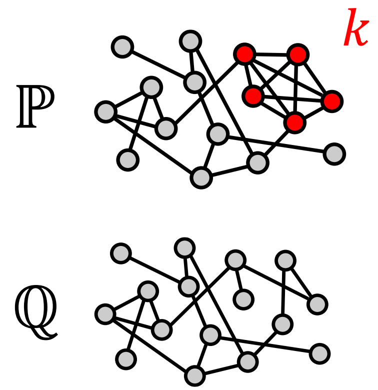

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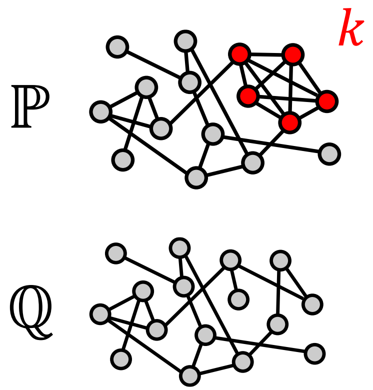

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  - Spectral methods; AMP; subgraph counts, e.g. triangle count $\sum_{i<j<\ell} Y_{ij} Y_{i\ell} Y_{j\ell}$

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- Idea arose from sum-of-squares, but can also be motivated directly
  [Barak, Hopkins, Kelner, Kothari, Moitra, Potechin ‘16]
  [Hopkins, Steurer ‘17]
  [Hopkins, Kothari, Potechin, Raghavendra, Schramm, Steurer ‘17]
  [Hopkins ‘18]

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  - The point is: degree-$O(\log n)$ polynomials capture important classes of poly-time algorithms, so if they fail, this rules out various approaches

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Theorem [Schramm, W '20] In the planted clique model,

- (Hard) If $k \le n^{1/2-\Omega(1)}$ then $\text{MMSE}_{\le O(\log n)} = (1 - o(1))\text{Var}(x)$
  - No better than the trivial degree-0 estimator $f(Y) = \mathrm{E}[x]$
- (Easy) If $k \ge n^{1/2+\Omega(1)}$ then $\text{MMSE}_{\le O(1)} = o(1/n)$
  - Small enough to guarantee exact recovery

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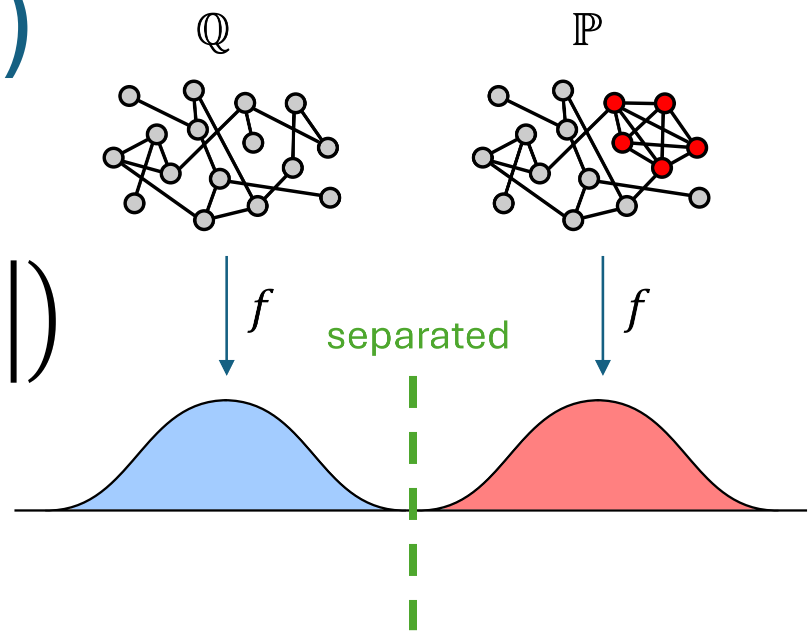

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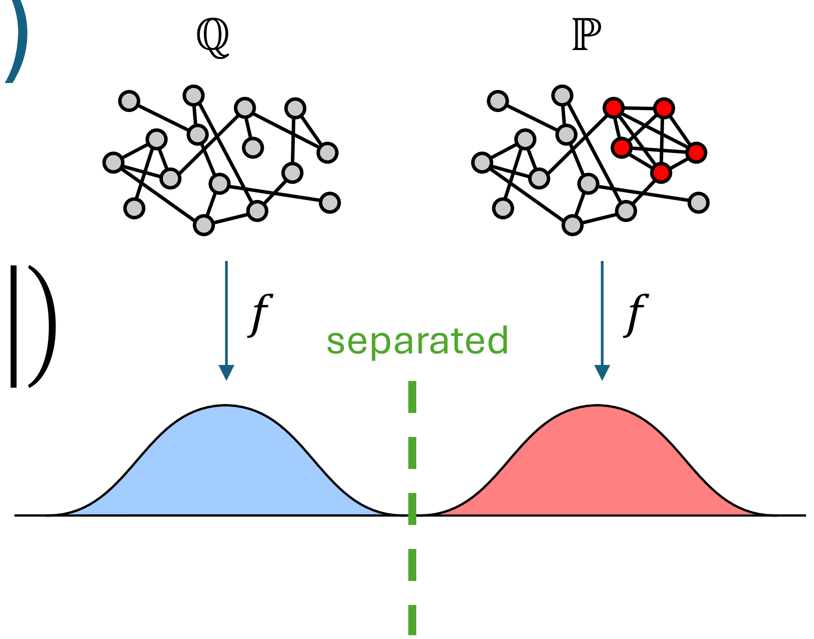

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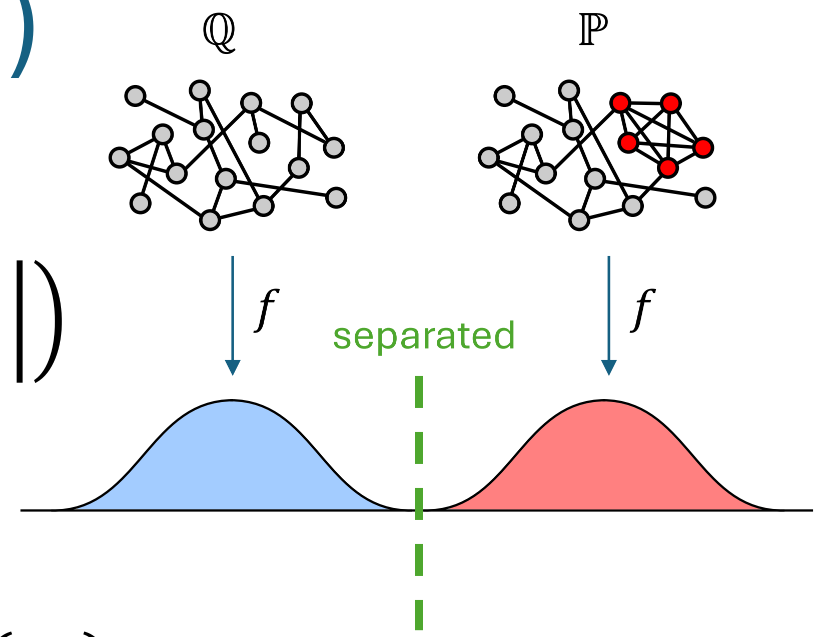

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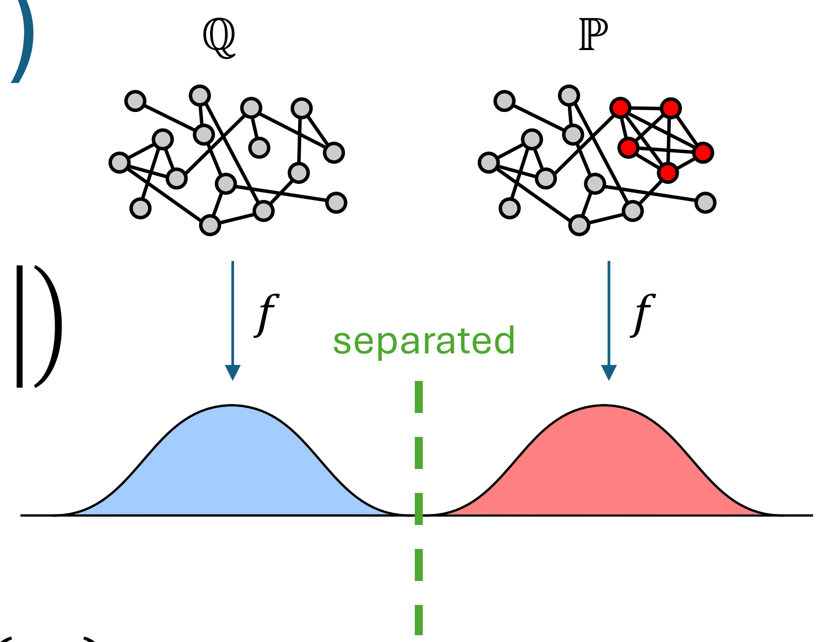

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Theorem [BHKKMP '16] In the planted clique model,

- (Hard) If $k \le n^{1/2-\Omega(1)}$, no degree-$O(\log n)$ polynomial strongly (or even weakly) separates $\mathbb{P}$ and $\mathbb{Q}$
- (Easy) If $k \ge n^{1/2+\Omega(1)}$, some degree-1 polynomial (edge count) strongly separates $\mathbb{P}$ and $\mathbb{Q}$

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    - $\mathrm{MMSE}_{\leq D}$ has the same flaw: even if you prove it’s large, you haven’t ruled out exact recovery by *thresholding* a polynomial

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- Heuristic for higher runtimes: degree-$n^\delta \approx$ time-$\exp(n^{\delta \pm o(1)})$

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  - sparse PCA, spiked Wigner/Wishart matrix, planted submatrix, group synchronization, …
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- *"Advantage"* a.k.a. *"norm of the low-degree likelihood ratio"* $\|L^{\leq D}\|$

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Fact: $\mathrm{Adv}_{\leq D} \coloneqq \sup_{f \deg D} \frac{\mathrm{E}_{\mathbb{P}}[f]}{\sqrt{\mathrm{E}_{\mathbb{Q}}[f^2]}} = \sqrt{\sum_i \mathrm{E}_{\mathbb{P}}[h_i]^2}$ where $\{h_i\}$ is a basis of *orthonormal polynomials* for $\mathbb{R}[Y]_{\leq D}$ under $\mathbb{Q}$: $\mathrm{E}_{Y\sim\mathbb{Q}}\left[h_i(Y)h_j(Y)\right] = \delta_{ij}$

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- Generally, can construct orthogonal polynomials when $\mathbb{Q}$ is a product measure (independent coordinates) $\mathbb{Q}$

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Sharp Thresholds in Estimation

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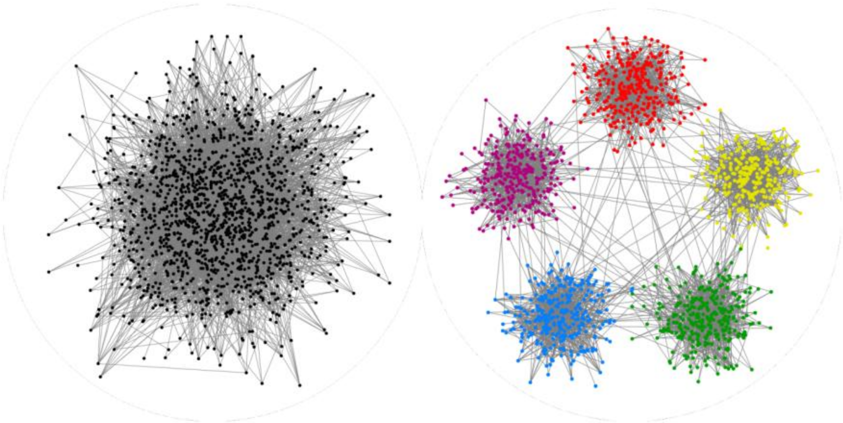

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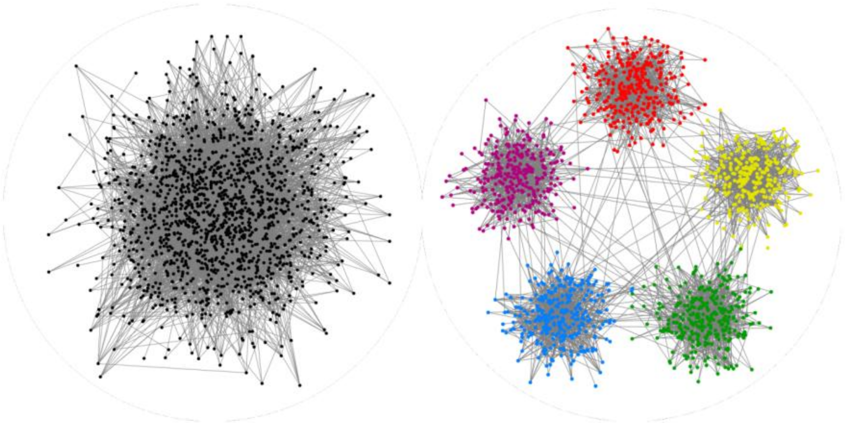

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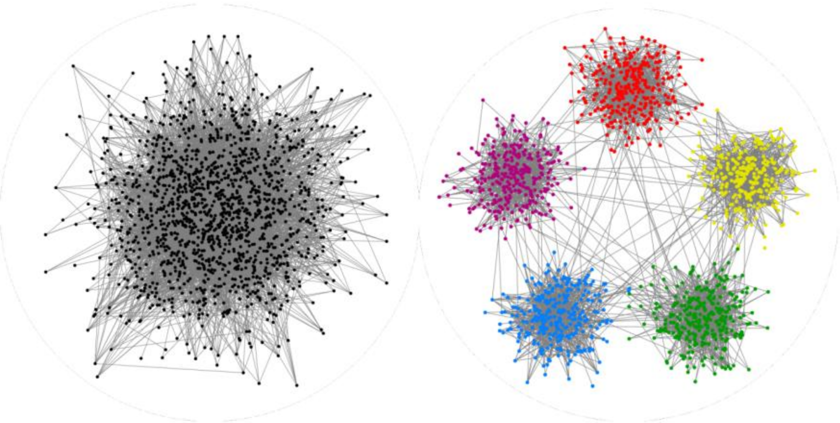

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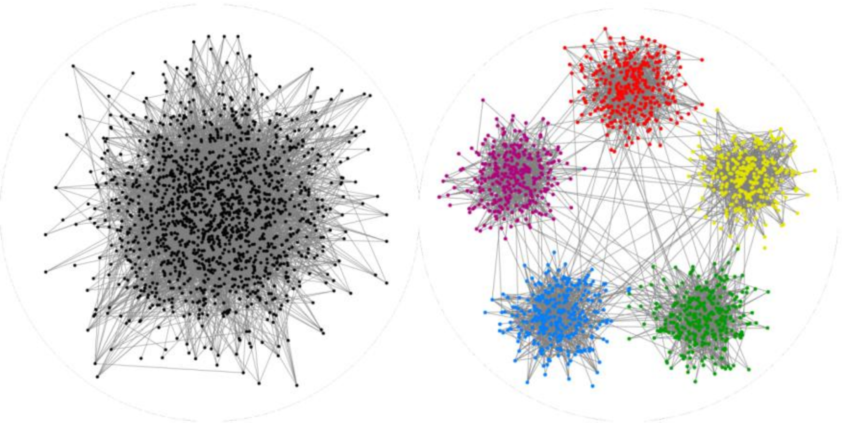

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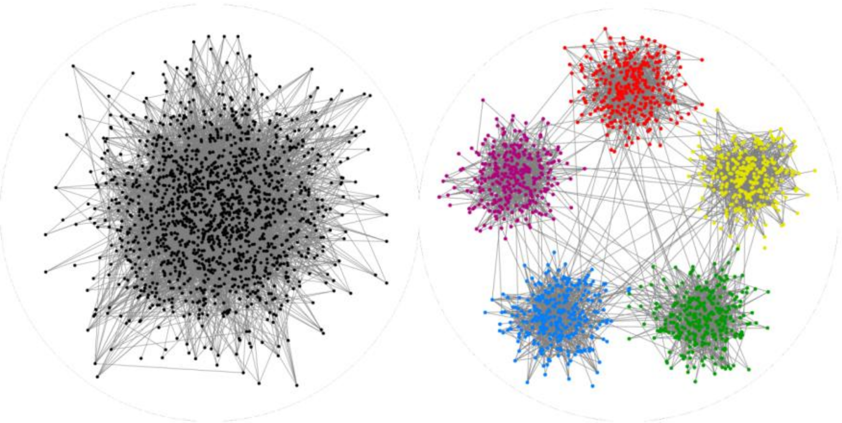

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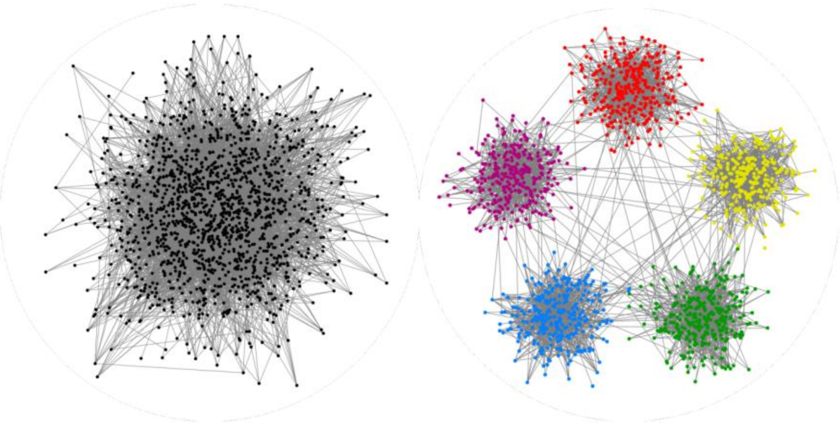

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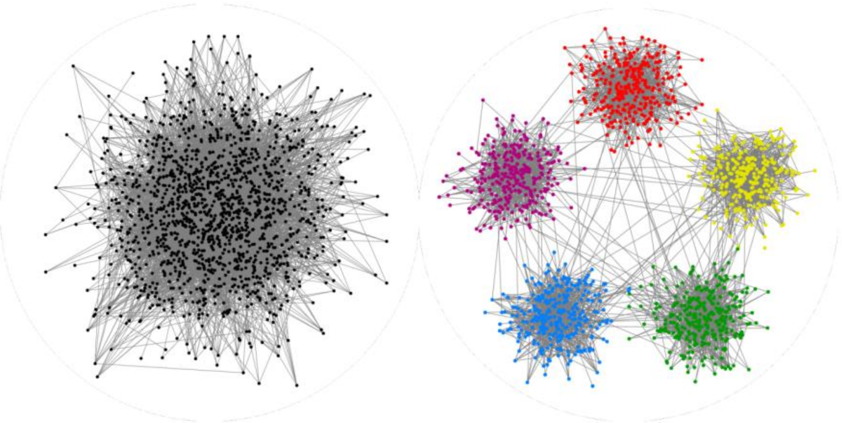

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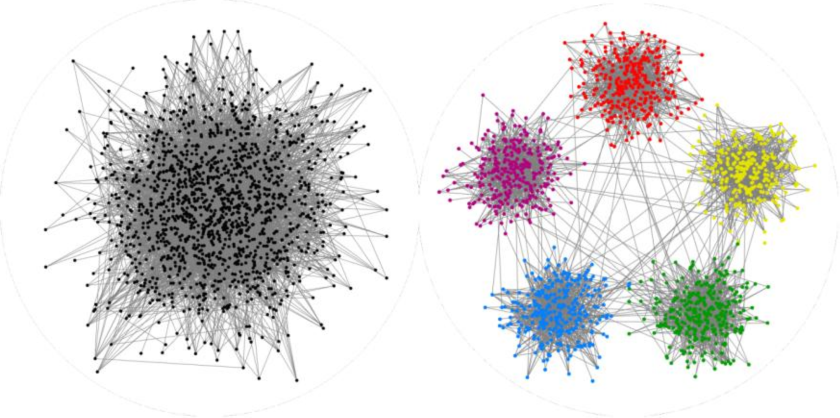

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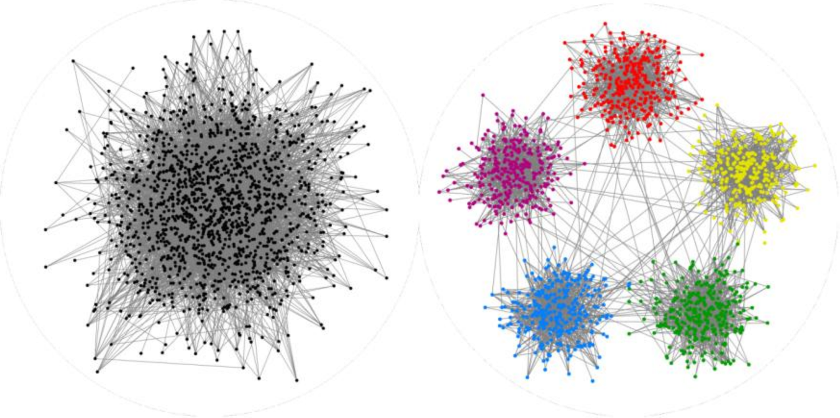

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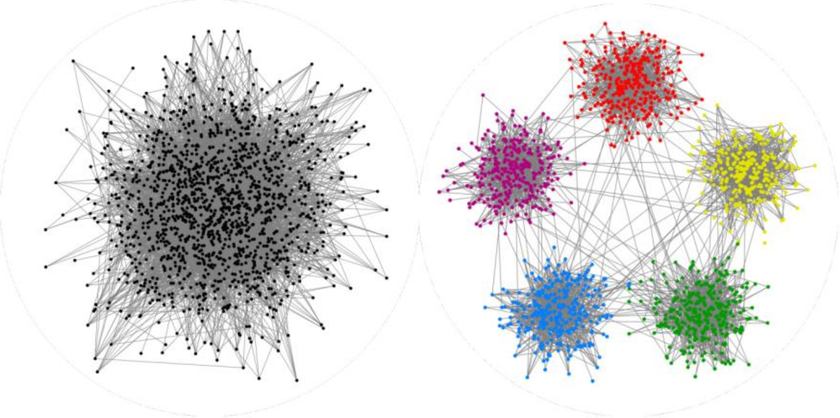

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  - No other frameworks apply here (?)

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- "Redemption"
  - Kikuchi hierarchy [W, Alaoui, Moore '19]
  - Averaged gradient descent [Biroli, Cammarota, Ricci-Tersenghi '19]
  - Modified MCMC [Lovig, Sheehan, Tsirkas, Zadik '25]
  - ... but somewhat problem-specific (?)

Tasks

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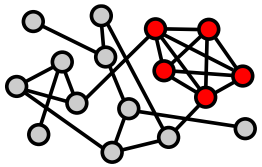
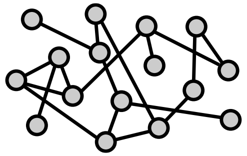
$\mathbb{P}$  $\mathbb{Q}$ 

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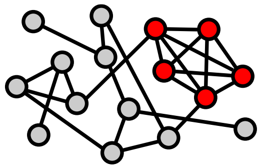
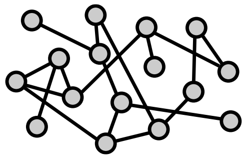
$\mathbb{P}$  $\mathbb{Q}$ 

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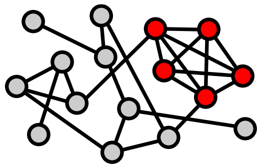

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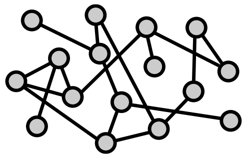

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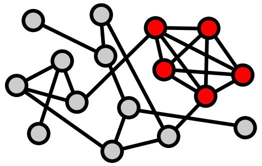
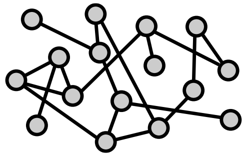
$\mathbb{P}$  $\mathbb{Q}$ 

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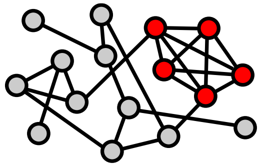
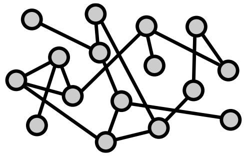
$\mathbb{P}$  $\mathbb{Q}$ 

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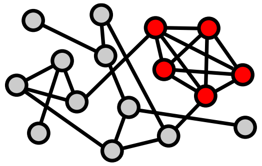
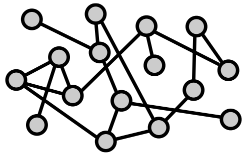
$\mathbb{P}$  $\mathbb{Q}$ 

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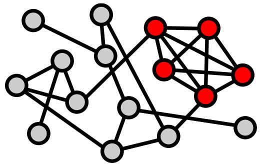
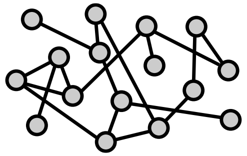
Detection ($\mathbb{P}$ vs $\mathbb{Q}$)

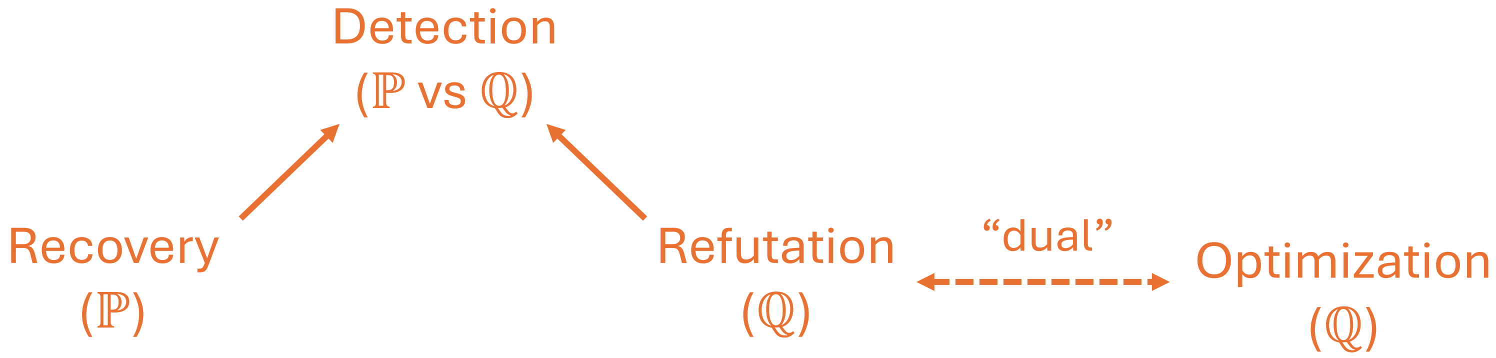
Recovery ($\mathbb{P}$)

Refutation ($\mathbb{Q}$) ← "dual" → Optimization ($\mathbb{Q}$)

Tasks

$\mathbb{P}$  $\mathbb{Q}$ 

- Using planted clique as a running example...
- **Detection (a.k.a. Testing)**: Decide if a given graph came from $\mathbb{P}$ or $\mathbb{Q}$
- **Recovery (a.k.a. Estimation)**: Given $G \sim \mathbb{P}$, find <u>the</u> planted clique
- **(Non-planted) Optimization**: Given $G \sim \mathbb{Q}$, find <u>any</u> k-clique
- **Refutation (or Certification)**: Given $G \sim \mathbb{Q}$, prove there's no k-clique



- These tasks can all have different thresholds in general

Frameworks vs Tasks

Which frameworks can give **hardness results** for which tasks?

	AMP	OGP	SOS	SQ	LD
Detection			✓	✓	✓
Recovery	✓	✓		✓	✓
Optimization	✓	✓			✓
Refutation			✓		✓

Known Connections

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Despite **many** caveats, some known connections among frameworks

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“Statistical Physics” / “Geometric”

Belief Propagation (BP)

Approximate Message Passing (AMP)

Overlap Gap Property (OGP)

Free Energy Barriers (Franz-Parisi Potential)

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"Computer Science" / "Algebraic"

Sum-of-Squares (SOS)

Spectral Methods

Low-Degree Polynomials

Statistical Query (SQ)

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Approximate Message Passing (AMP)

[GJ'19]

Overlap Gap Property (OGP)

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Gamarnik, Jagannath '19

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↓

Approximate Message Passing (AMP)

↑ [GJ'19]

Overlap Gap Property (OGP)

Free Energy Barriers (Franz-Parisi Potential)

"Computer Science" / "Algebraic"

Sum-of-Squares (SOS)

↓ [HKPRSS'17]

Spectral Methods

↓ Power Iter

Low-Degree Polynomials

Statistical Query (SQ)

Hopkins, Kothari, Potechin, Raghavendra, Schramm, Steurer '17

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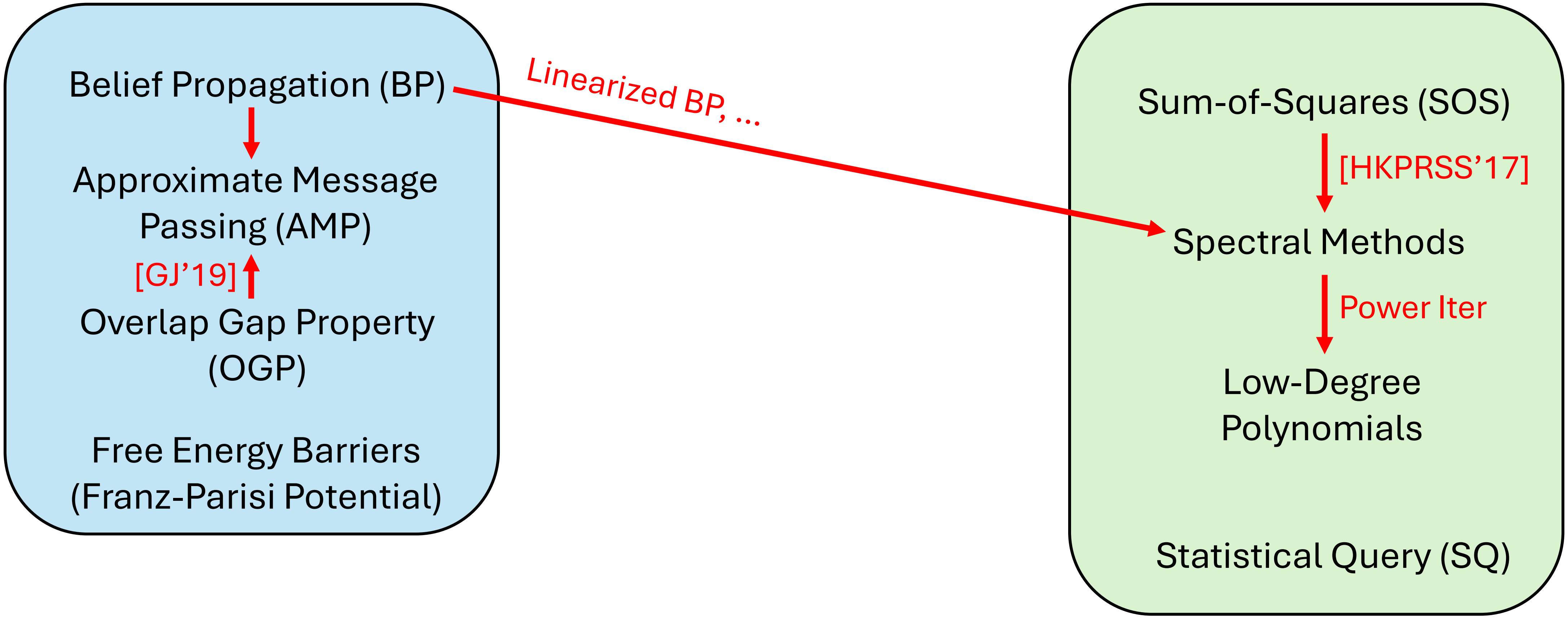

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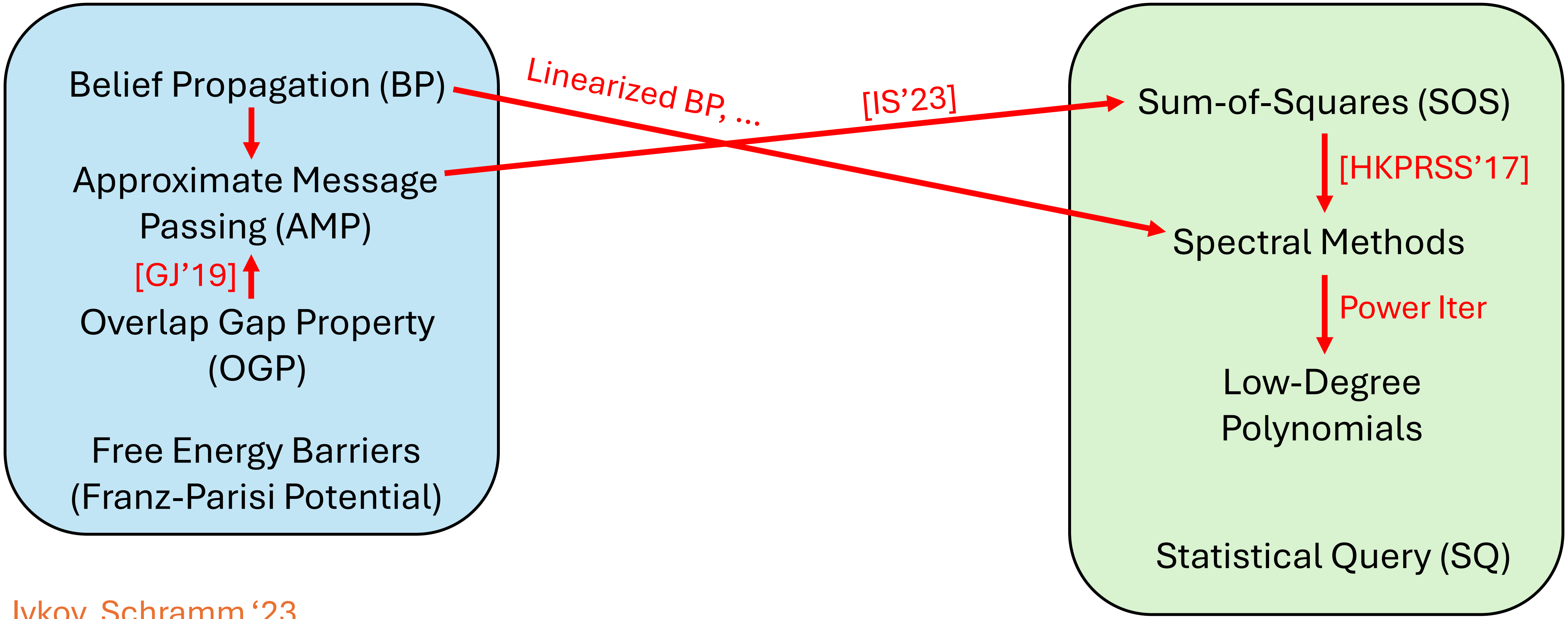

Ivkov, Schramm ‘23

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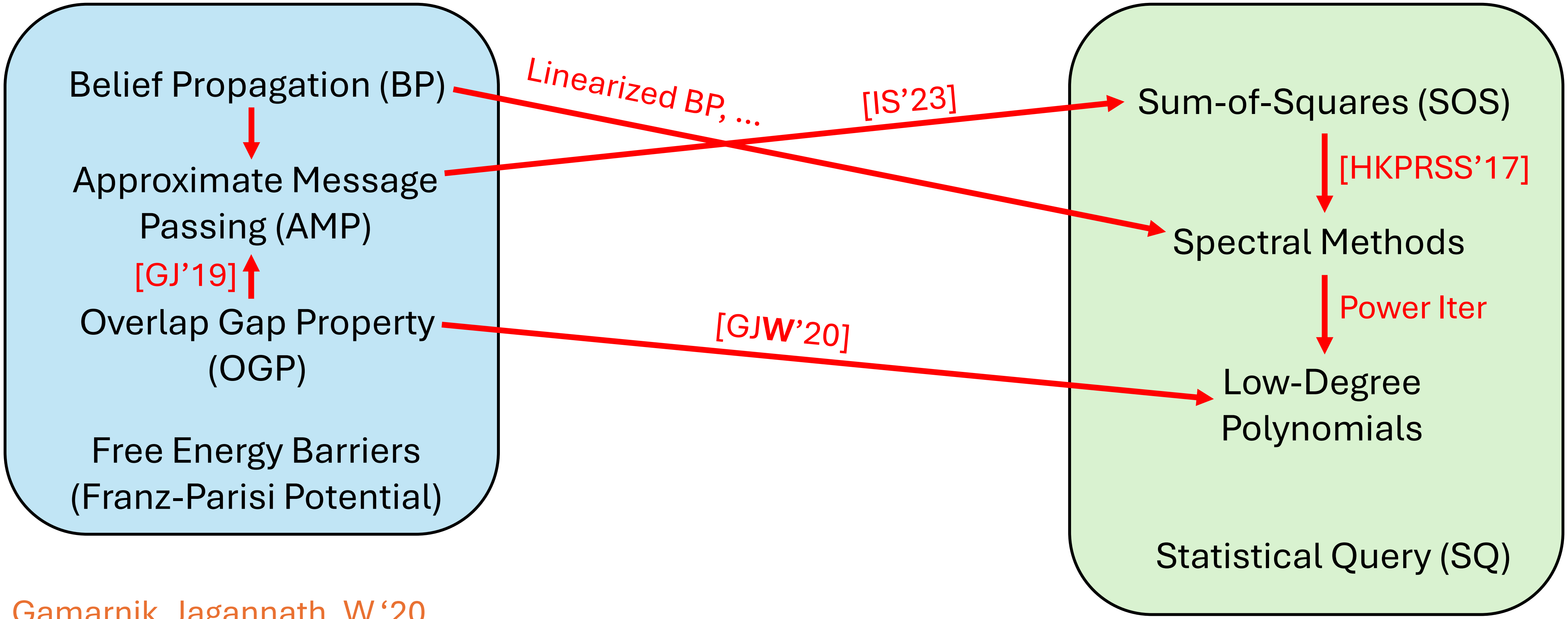
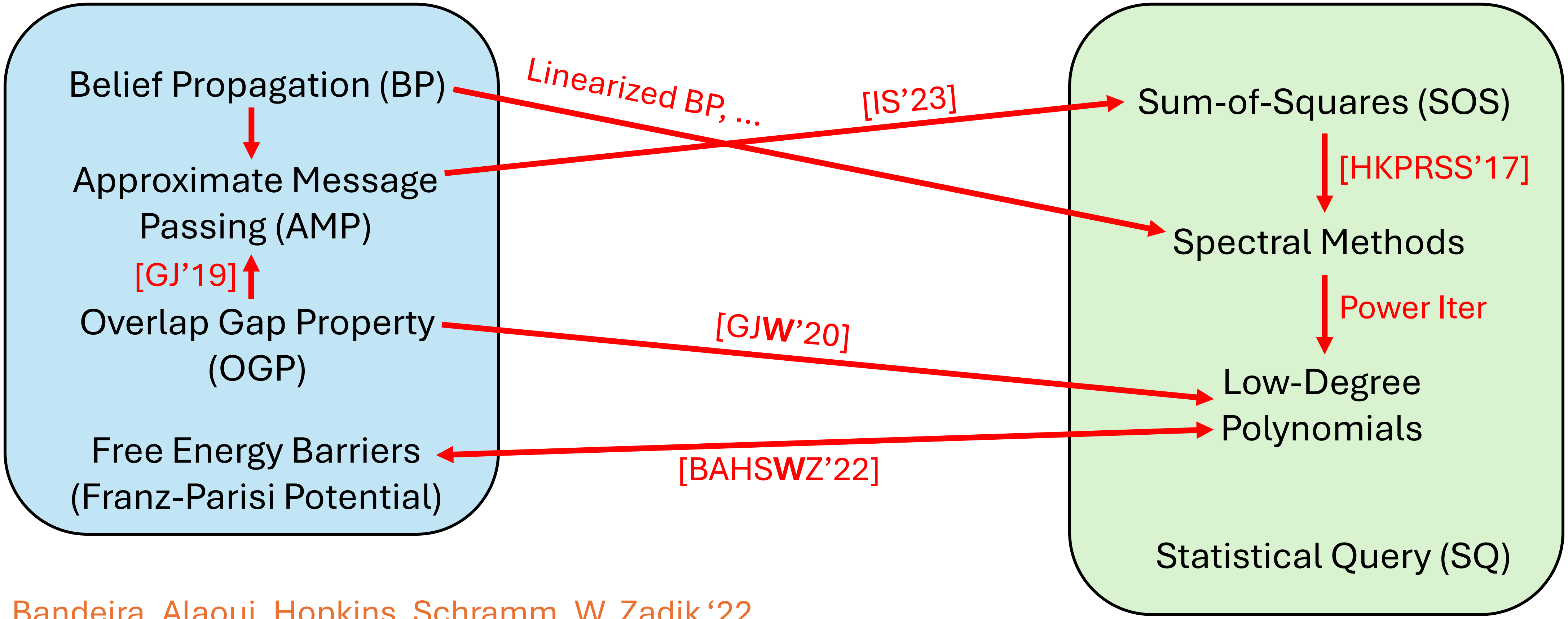


Gamarnik, Jagannath, W '20

Known Connections

Despite **many** caveats, some known connections among frameworks

“Statistical Physics” / “Geometric” **“Computer Science” / “Algebraic”**



Bandeira, Alaoui, Hopkins, Schramm, W, Zadik ‘22

Known Connections

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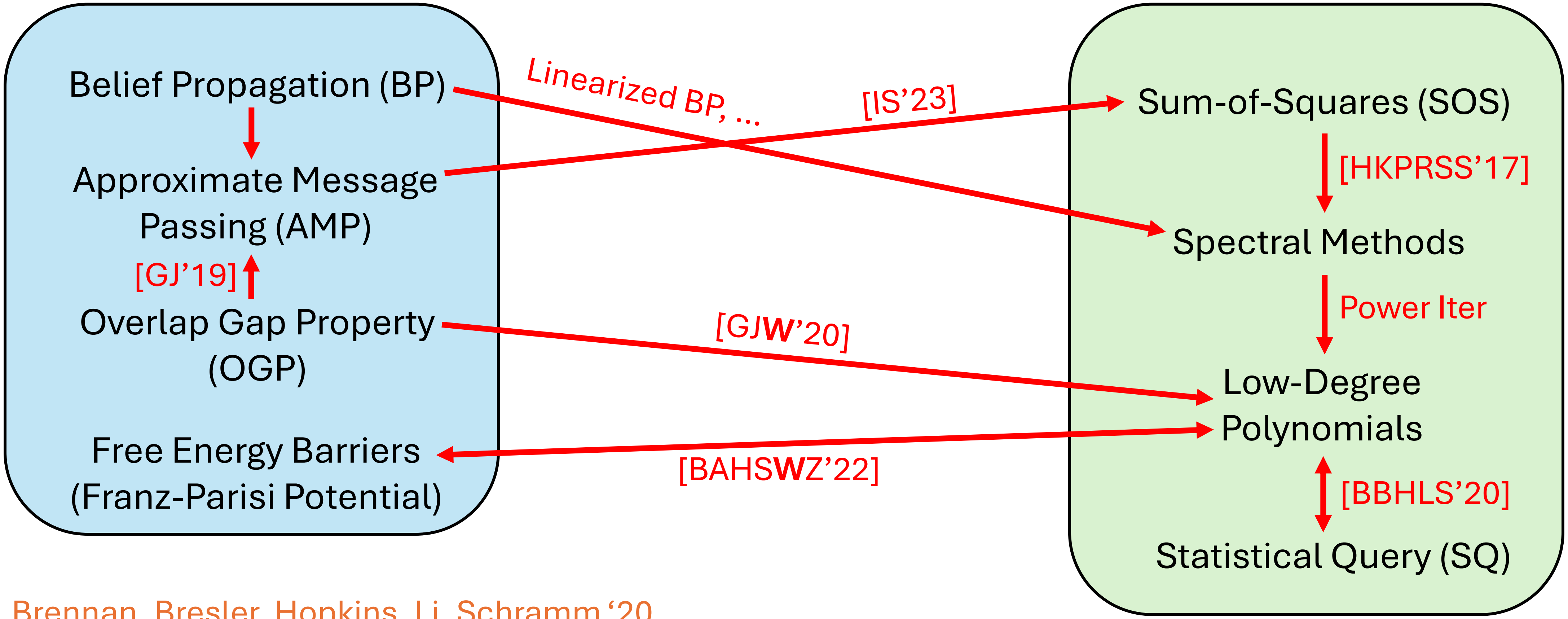
"Statistical Physics" / "Geometric" **"Computer Science" / "Algebraic"**



Brennan, Bresler, Hopkins, Li, Schramm '20

Known Connections

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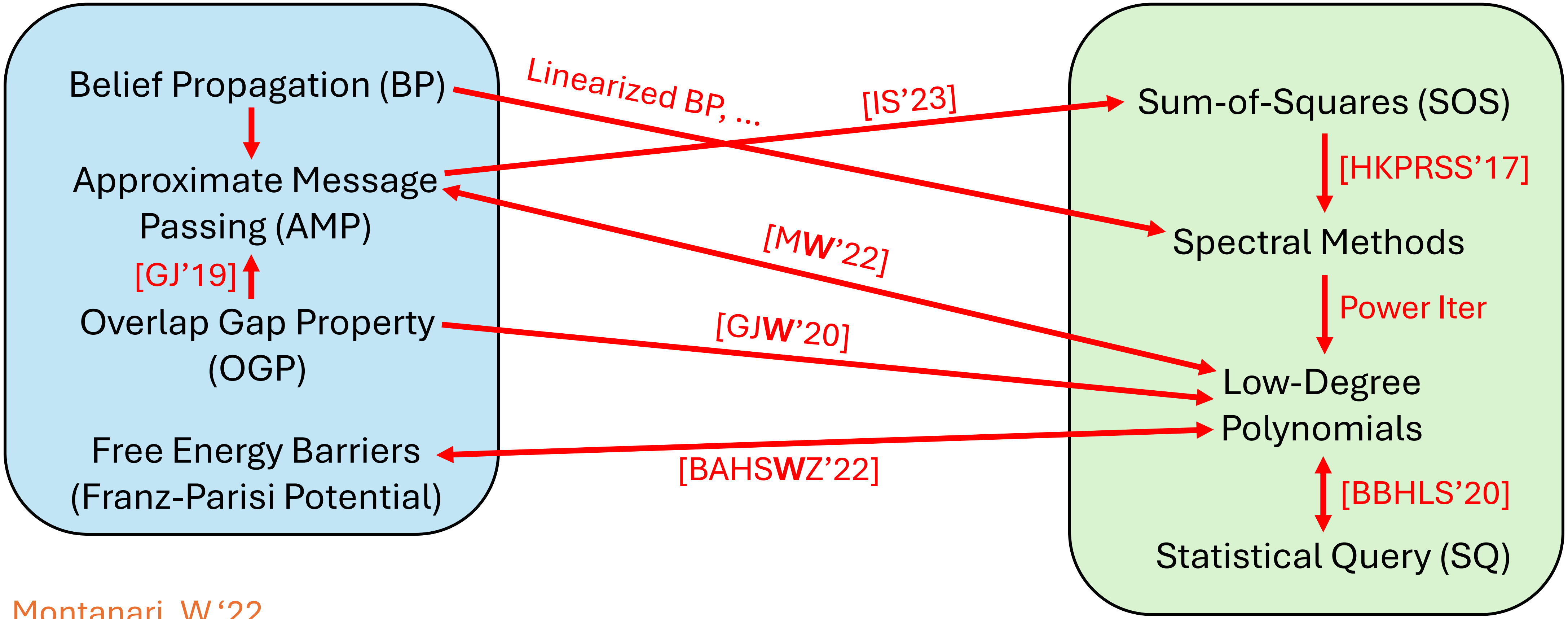

Montanari, W ‘22

Known Connections

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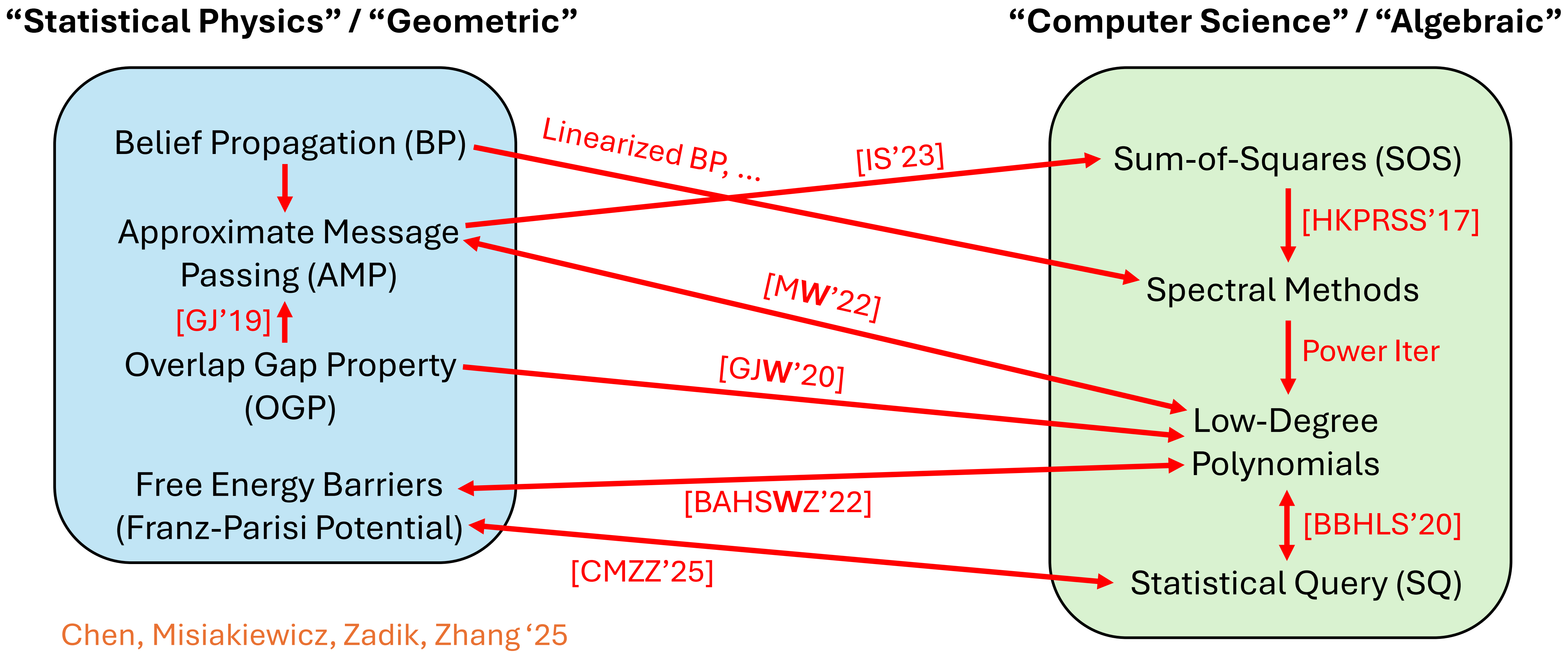

Chen, Misiakiewicz, Zadik, Zhang ‘25

Known Connections

Despite **many** caveats, some known connections among frameworks