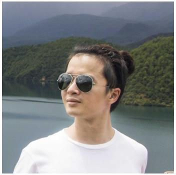
# Understanding Statistical-vs-Computational Tradeoffs via the Low-Degree Likelihood Ratio


Alex Wein
Courant Institute, NYU


Joint work with:




Afonso Bandeira
(ETH Zurich)

Yunzi Ding
(NYU)

Tim Kunisky
(NYU)

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Goal: develop a theory to understand which statistical tasks can be solved efficiently (and which ones cannot)

# Part I: Statistical-to-Computational Gaps and the "Low-Degree Method"

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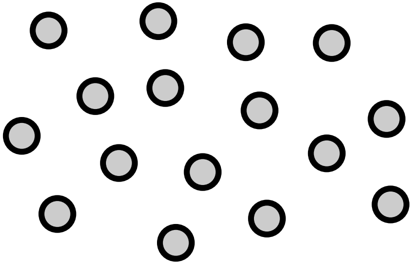

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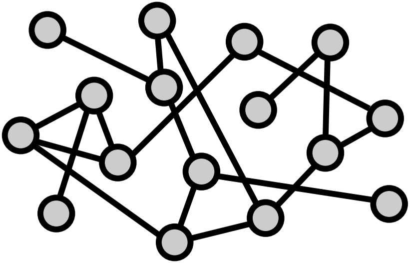

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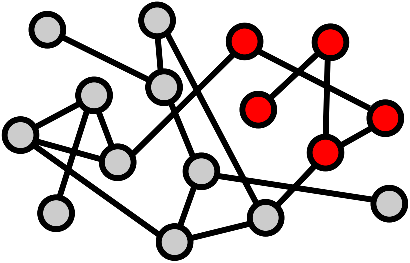

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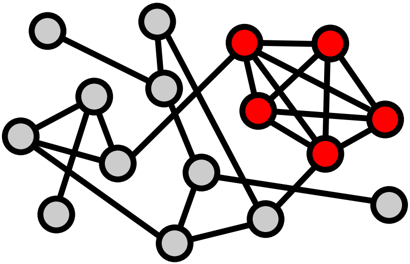

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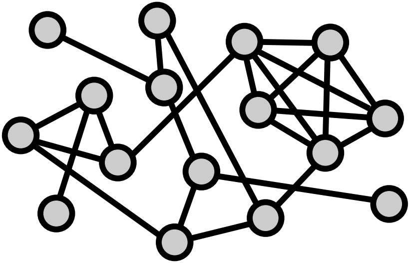

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  - Goal: find the clique

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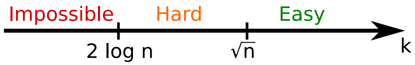


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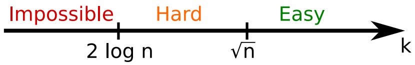


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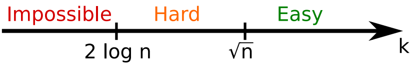


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Q: What fundamentally makes a problem easy or hard?

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- This talk: "low-degree method"

  [Barak, Hopkins, Kelner, Kothari, Moitra, Potechin '16; Hopkins, Steurer '17;

  Hopkins, Kothari, Potechin, Raghavendra, Schramm, Steurer '17; Hopkins '18 (PhD thesis)]

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Compute $\max_{f \text{ deg } D} \dfrac{\mathbb{E}_{Y \sim \mathbb{P}}[f(Y)]}{\sqrt{\mathbb{E}_{Y \sim \mathbb{Q}}[f(Y)^2]}}$ $\dfrac{\text{mean in } \mathbb{P}}{\text{fluctuations in } \mathbb{Q}}$

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Norm of low-degree likelihood ratio

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Conjecture (informal variant of [Hopkins '18])

*For "nice" $\mathbb{Q}, \mathbb{P}$, if $\|L^{\leq D}\| = O(1)$ for some $D = \omega(\log n)$ then no polynomial-time algorithm can distinguish $\mathbb{Q}, \mathbb{P}$ with success probability $1 - o(1)$.*

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Degree-$O(\log n)$ polynomials $\Leftrightarrow$ Polynomial-time algorithms

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The case $D = \infty$: If $\|L\| = O(1)$ (as $n \to \infty$) then no test can distinguish $\mathbb{Q}$ from $\mathbb{P}$ (with success probability $1 - o(1)$)

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- Spectral methods are believed to be as powerful as sum-of-squares for average-case problems [HKPRSS '17]

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# How to Compute $\|L^{\leq D}\|$

Additive Gaussian noise: $\mathbb{P}: Y = X + Z \quad$ vs $\quad \mathbb{Q}: Y = Z$
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Result: $\|L^{\leq D}\|^2 = \sum_{d=0}^{D} \frac{1}{d!} \mathbb{E}_{X,X'}[\langle X, X'\rangle^d]$

# References

For more on the low-degree method...

- Samuel B. Hopkins, PhD thesis '18: "Statistical Inference and the Sum of Squares Method"
  - Connection to SoS
- Survey article: Kunisky, W, Bandeira, "Notes on Computational Hardness of Hypothesis Testing: Predictions using the Low-Degree Likelihood Ratio", *arxiv:1907.11636*

# Part II: Sparse PCA

Based on: Ding, Kunisky, W., Bandeira, "Subexponential-Time Algorithms for Sparse PCA", *arxiv:1907.11635*

# Spiked Wigner Model

Observe $n \times n$ matrix $Y = \lambda x x^T + W$
Signal: $x \in \mathbb{R}^n$, $\|x\| = 1$
Noise: $W \in \mathbb{R}^{n \times n}$ with entries $W_{ij} = W_{ji} \sim \mathcal{N}(0, 1/n)$ i.i.d.
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Sharp threshold: PCA can detect and recover the signal iff $\lambda > 1$

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But what if $x$ is sparse?

# Sparse PCA

Suppose $x \in \mathbb{R}^n$ is drawn from the $k$-sparse Rademacher prior:

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Assume $\lambda < 1$ is a constant

- PCA fails

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Runtime: $\binom{n}{k} \approx n^k \approx \exp(k)$

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Variant: covariance thresholding is poly-time and succeeds when $k \lesssim \sqrt{n}$ (removes log factor) [Krauthgamer, Nadler, Vilenchik '15, Deshpande, Montanari '14]

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Question: exactly how hard is the "hard" regime?

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*Suppose* $\lambda = \Theta(1)$.

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And indeed we will find such an algorithm...

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Runtime: $\binom{n}{\ell} \approx n^\ell \approx \exp(\ell)$

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For any given $k$, choose $\ell \approx k^2/n$, get runtime $\exp(k^2/n)$

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Technically, need independent copies of $Y$ for steps 1 & 2

- $Y + W'$ and $Y - W'$ where $W'$ is independent copy of $W$

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Thanks!