Computational Barriers to Estimation from Low-Degree Polynomials

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Joint work with:

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Stanford

Part I: Why Low-Degree Polynomials?

Problems in High-Dimensional Statistics

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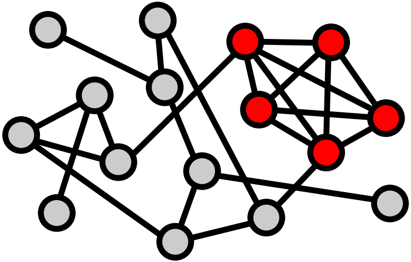

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What makes problems easy vs hard?

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Today: self-contained motivation (without SoS)

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- Or any of the above applied to $\tilde{Y} = g(Y)$ — $\deg g = O(1)$

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Caveat: Gaussian elimination for planted XOR-SAT

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  - Need to argue that starting problem is hard [BB20]

Part II: Detection

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Goal: hypothesis test with error probability $o(1)$ between:

- Null model $Y \sim \mathbb{Q}_n$ e.g. $G(n, 1/2)$
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$$= \begin{cases} \omega(1) & \text{"degree-}D\text{ polynomial succeed"} \\ O(1) & \text{"degree-}D\text{ polynomials fail"} \end{cases}$$

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Extended low-degree conjecture [Hopkins '18]:

degree-D polynomials $\Leftrightarrow$ $n^{\tilde{\Theta}(D)}$-time algorithms

$D = n^\delta \quad \Leftrightarrow \quad \exp(n^{\delta \pm o(1)})$ time

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<u>Numerator</u>: $\mathbb{E}_{Y\sim\mathbb{P}}[f(Y)]$

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Proof: $\hat{L}_S = \mathbb{E}_{Y \sim \mathbb{Q}}[L(Y)Y^S] = \mathbb{E}_{Y \sim \mathbb{P}}[Y^S] \qquad \hat{f}^*_S = \mathbb{E}_{Y \sim \mathbb{P}}[Y^S]\,\mathbb{1}_{|S| \leq D}$

Part III: Recovery

Planted Submatrix

Example (planted submatrix): observe $n \times n$ matrix $Y = X + Z$

- Signal: $X = \lambda v v^\top \quad \lambda > 0 \quad v_i \sim \mathrm{Bernoulli}(\rho)$
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This closes detection-recovery gap partially but not all the way

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Equivalent to low-degree maximum correlation:

$$\mathrm{Corr}_{\le D} = \max_{f \text{ deg } D} \frac{\mathbb{E}[f(Y) \cdot v_1]}{\sqrt{\mathbb{E}[f(Y)^2]}}$$

Fact: $\mathrm{MMSE}_{\le D} = \mathbb{E}[v_1^2] - \mathrm{Corr}_{\le D}^2$

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- Implications for other algorithms?
  - E.g. convex programming, MCMC

References

- Detection (survey article)
  Notes on Computational Hardness of Hypothesis Testing: Predictions using the Low-Degree Likelihood Ratio
  Kunisky, W., Bandeira
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  Schramm, W.
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  Gamarnik, Jagannath, W.
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(extra scratch paper)