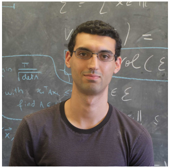
The Kikuchi Hierarchy and Tensor PCA

Alex Wein
Courant Institute, NYU

Joint work with:



Ahmed El Alaoui
Stanford

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Santa Fe Institute

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This theory has been hugely successful at precisely understanding statistical and computational limits of many problems.

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Meta-question: unify the statistical physics and SoS approaches?

This talk: case study on tensor PCA – a problem where statistical physics and SoS disagree (!!!)

Tensor PCA (Principal Component Analysis)

Definition (Spiked Tensor Model [Richard-Montanari '14])

$x \in \{\pm 1\}^n$ – signal
$p \in \{2, 3, 4, \dots\}$ – tensor order
For each subset $U \subseteq [n]$ of size $|U| = p$, observe

$$Y_U = \lambda \prod_{i \in U} x_i + \mathcal{N}(0, 1)$$

$\lambda \geq 0$ – signal-to-noise parameter
Goal: given $\{Y_U\}$, recover x (with high probability as $n \to \infty$)

- "For every p variables, get a noisy observation of their parity"
- In tensor notation: $Y = \lambda x^{\otimes p} + Z$ where Z is symmetric noise
- Case $p = 2$ is the spiked Wigner matrix model $Y = \lambda x x^\top + Z$

Algorithms for Tensor PCA

Maximum likelihood estimation (MLE):

$$\Pr[x|Y] \propto \exp\left(\sum_{|U|=p} \lambda Y_U \prod_{i \in U} x_i\right) = \exp\left(\frac{\lambda}{p}\langle Y, x^{\otimes p}\rangle\right)$$

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- Problem: requires exponential time 2^n

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These only succeed when $\lambda \gg n^{-1/2}$

- Recall: MLE works for $\lambda \sim n^{(1-p)/2}$

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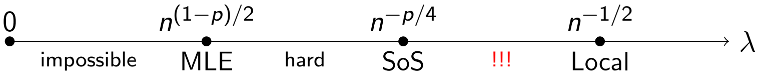

Local algorithms (gradient descent, AMP, ...) are suboptimal when $p \geq 3$

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Interpolates between SoS and MLE:

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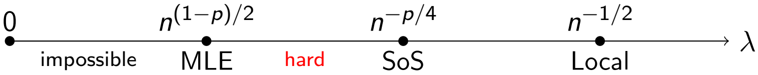

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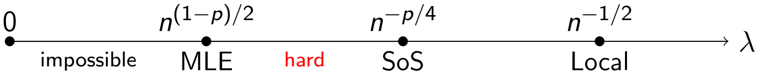

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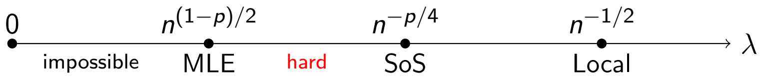

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For "soft" thresholds (like tensor PCA): BP/AMP can't be optimal

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For more, see the survey Kunisky-W.-Bandeira, “Notes on Computational Hardness of Hypothesis Testing: Predictions using the Low-Degree Likelihood Ratio”, arXiv:1907.11636

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Motivating the Algorithm: Belief Propagation / AMP

General setup: unknown signal $x \in \{\pm 1\}^n$, observed data Y

Want to understand posterior $\Pr[x|Y]$

Find distribution μ over $\{\pm 1\}^n$ minimizing free energy
$\mathcal{F}(\mu) = \mathcal{E}(\mu) - \mathcal{S}(\mu)$

- "Energy" and "entropy" terms
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- We will use a spectral method based on the Kikuchi Hessian

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Our approach: Kikuchi Hessian

- Bottom eigenvector of Hessian of $\mathcal{K}(m)$ with respect to moments $m = \{m_i, m_{ij}, \ldots\}$

The Algorithm

Definition (Symmetric Difference Matrix)

Input: an order-p tensor $Y = (Y_U)_{|U|=p}$ (with p even) and an integer ℓ in the range $p/2 \leq \ell \leq n - p/2$. Define the $\binom{n}{\ell} \times \binom{n}{\ell}$ matrix (indexed by ℓ-subsets of $[n]$)

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This is a message-passing algorithm among sets of size ℓ

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Simplest statistical task: detection

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Let $M = \sum_i z_i A_i$ where $z_i \sim \mathcal{N}(0,1)$ independently and $\{A_i\}$ is a finite sequence of fixed symmetric $d \times d$ matrices. Then, for all $t \geq 0$,

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In our case, $\sum_i (A_i)^2$ is a multiple of the identity

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SoS approach: given noise tensor Y, want to certify (prove) an upper bound on tensor injective norm

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- [Biroli, Cammarota, Ricci-Tersenghi '19, "How to iron out rough landscapes and get optimal performances"]
  - A different form of "redemption" for local algorithms
  - Replicated gradient descent

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  - Systematically obtain optimal spectral methods in general?

Summary

- Local algorithms are suboptimal for tensor PCA
  - E.g. gradient descent, AMP
  - Keep track of an n-dimensional state
  - Nearly-linear runtime
- Why suboptimal?
  - Soft threshold: optimal algorithm cannot be nearly-linear time
  - For p-way data, need p-way algorithm?
- "Redemption" for local algorithms and AMP
  - Hierarchy of message-passing algorithms: symm. diff. matrices
  - Keep track of beliefs about higher-order correlations
  - Minimize Kikuchi free energy
  - Matches SoS (conjectured optimal)
  - Proof is much simpler than prior work
- Future directions
  - Unify statistical physics and SoS?
  - Systematically obtain optimal spectral methods in general?

Thanks!